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Intrinsic Josephson effect and nonequilibrium soliton structures in two-gap superconductors

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We predict a new dynamic state in current-carrying superconductors with multicomponent order parameter. If the current density J exceeds a critical value J_t, an interband breakdown caused by charge imbalance of nonequilibrium quasiparticles occurs. For $J > J_t$, the electric field penetrating from current leads gives rise to various static and dynamic soliton phase textures, and voltage oscillations similar to the nonstationary Josephson effect. We propose experiments to observe these effects which would probe the multicomponent nature of the superconducting order parameter.

There are experimental evidences that several superconductors, including MgB_2 [1, 2], $NbSe_2$ [3], the heavy-fermion UPt_3 [4], the organic $(TMTSF)_2X$ [5] and $\kappa - BEDT$ [6] compounds may have a multicomponent order parameter ψ with internal degrees of freedom [7, 8]. For instance, for two weakly coupled s-wave order parameters $\psi_1 = \Delta_1 e^{i\theta_1}$ and $\psi_2 = \Delta_2 e^{i\theta_2}$ corresponding to two different parts of the Fermi surface (like in MgB_2), the internal degree of freedom is the interband phase difference $\theta(\mathbf{r}, t) = \theta_1 - \theta_2$. In this case, besides the usual phase-locked states ($\theta = 0$ or π), intrinsic phase textures $\theta(\mathbf{r}, t)$ can occur. The soft modes associated with the fluctuations of $\theta(\mathbf{r}, t)$ are nearly decoupled from the gap fluctuations and behave like the Anderson plasmons in Josephson junctions [7]. These modes may manifest themselves as resonances in the ac Josephson effect [9], or static 2π phase kinks in $\theta(x)$ [10].

In this Letter we show that, although static metastable textures $\theta(\mathbf{r})$ do not manifest themselves in the equilibrium magnetic response, the situation radically changes for *nonequilibrium* current states where the charge imbalance near normal leads gives rise to dynamic phase slip structures $\theta(\mathbf{r}, t)$ propagating into a superconductor (Fig. 1). We predict a new dynamic state above the critical current density $J > J_t$ which marks the onset of the current-induced breakdown of the superconducting state. For the multicomponent ψ, such breakdown is a two-stage process. First, at $J = J_t$, an interband phase breakdown occurs, resulting in spontaneous phase solitons in $\theta(x, t)$ and ac voltage oscillations. At higher currents, $J = J_d$, both gaps $\Delta_{1,2}$ are suppressed by the pairbreaking effects. For weak interband coupling, J_t is much smaller than the depairing current density J_d.

We obtain the equations of motion for θ and the electric field $\mathbf{E}$ near the critical temperature $T \approx T_c$, using the time-dependent Ginzburg-Landau (TDGL) equations [11] generalized to a two-gap superconductor:

$$\Gamma_\mu(\partial_t \psi_\mu - 2\pi ci\varphi/\phi_0) = -\delta F/\delta\psi_\mu, \quad (1)$$

where μ runs from 1 to 2, φ is the electric potential, ϕ_0 is the flux quantum, c is the speed of light, Γ_μ are damping constants, and the free energy $F = \int d^3\mathbf{r}(f_1 + f_2 + f_m + f_{int})$ contains the magnetic part $f_m = |\nabla \times \mathbf{A}|^2/8\pi$, the GL intraband part f_μ, and the interband interaction f_{int}

$$f_\mu = \alpha_\mu|\psi_\mu|^2 + \frac{\beta_\mu}{2}|\psi_\mu|^4 + g_\mu\left|\left(\nabla + \frac{2\pi i}{\phi_0}\mathbf{A}\right)\psi_\mu\right|^2, \quad (2)$$

$$f_{int} = \gamma(\psi_1\psi_2^* + \psi_1^*\psi_2) = 2\gamma\Delta_1\Delta_2\cos\theta. \quad (3)$$

Here $\mathbf{A}$ is the vector potential, γ is a coupling constant, the asterisk means complex conjugation. The qualitative results of this work remain valid for any other periodic dependencies of $f_{int}(\theta)$ on θ, for example, $f_{int} = 2\Delta_1^2\Delta_2^2(\gamma_1\cos 2\theta + \gamma_2)$, suggested for heavy-fermion superconductors [4]. We focus here on two-gap superconductors for which two cases are possible: (i) The Cooper instability occurs at the same temperature in both bands, thus $\alpha_\mu(T) \propto T - T_c$, $\gamma \propto T - T_c$, while g_μ, and β_μ are constants. (ii) The critical temperature T_{c1} in the band 1 is higher than T_{c2}, in the band 2, so ψ_1 appears spontaneously below T_c, while ψ_2 at $T_{c1} < T < T_c$ is induced by the interband interaction, because $\alpha_1 \propto T - T_{c1}$, $\alpha_2 \propto T - T_{c2}$, and γ, g_μ and β_μ are constants [12]. In this paper we consider the first scenario, assuming that $\gamma \ll \alpha_{1,2}$, which is likely to occur in MgB_2 [1, 2].

Eqs. (1) yield the following gap equations

$$\Gamma_\mu\dot{\Delta}_\mu = g_\mu\nabla^2\Delta_\mu - \tilde{\alpha}_\mu\Delta_\mu - \beta_\mu\Delta_\mu^3 - \gamma\Delta_{\overline{\mu}}\cos\theta, \quad (4)$$

where $\overline{\mu} = 2$ if $\mu = 1$, and $\overline{\mu} = 1$ if $\mu = 2$, $\theta = \theta_1 - \theta_2$, $\tilde{\alpha}_\mu = \alpha_\mu + (2\pi Q_\mu/\phi_0)^2 g_\mu$, and $\mathbf{Q}_\mu = \mathbf{A} - \phi_0\nabla\theta_\mu/2\pi$. The imaginary part of Eq. (1) gives

$$\Gamma_\mu\left(\dot{\theta}_\mu - \frac{2\pi c}{\phi_0}\varphi\right)\Delta_\mu^2 = -\frac{2\pi g_\mu}{\phi_0}\nabla(\Delta_\mu^2\mathbf{Q}_\mu) \pm \gamma\Delta_1\Delta_2\sin\theta \quad (5)$$

$$\nabla \times \nabla \times \mathbf{A} = (4\pi/c)(\sigma\mathbf{E} + \mathbf{J}_s) \quad (6)$$

with the plus sign corresponding to $\mu = 1$. The current density $\mathbf{J} = \sigma\mathbf{E} + \mathbf{J}_s$ has the ohmic component $\mathbf{J}_n = \sigma\mathbf{E}$

proportional to the electric field $\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}/c$, and the normal conductivity σ. The supercurrent

$$\mathbf{J}_s = -8\pi^2 c(g_1\Delta_1^2\mathbf{Q}_1 + g_2\Delta_2^2\mathbf{Q}_2)/\phi_0^2 \qquad (7)$$

is a sum of independent intraband contributions. The boundary conditions, $(i\partial_n + 2\pi A_n/\phi_0)\psi_\mu = i\psi_\mu/l_\mu$, between a superconductor and a normal metal imply zero perpendicular supercurrents for *both* ψ_1 and ψ_2. Excluding $\mathbf{Q}_{1,2} = -(\phi_0\mathbf{J}_s \pm 4\pi c g_{2,1}\Delta_{2,1}^2\nabla\theta)\phi_0/[8\pi^2 c(g_1\Delta_1^2 + g_2\Delta_2^2)]$ from Eqs. (2) and (7), we obtain the energy density $f = f_\theta + f_e + \sum_\mu[\alpha_\mu\Delta_\mu^2 + g_\mu(\nabla\Delta_\mu)^2 + \beta_\mu\Delta_\mu^4/2]$, where the magnetic and phase dependent parts, f_m and f_θ turn out to be decoupled from each other:

$$f_m = \frac{H^2}{8\pi} + \frac{\phi_0^2 J_s^2}{(4\pi c)^2(g_1\Delta_1^2 + g_2\Delta_2^2)} \qquad (8)$$

$$f_\theta = \frac{g_1 g_2\Delta_1^2\Delta_2^2(\nabla\theta)^2}{(g_1\Delta_1^2 + g_2\Delta_2^2)} + 2\gamma\Delta_1\Delta_2\cos\theta \qquad (9)$$

For constant gaps Δ_μ, variation of Eq. (8) with respect to $\mathbf{H}$ yields the London equation $\lambda^2\nabla^2\mathbf{H} = \mathbf{H}$ with the magnetic penetration depth $\lambda^2 = \phi_0^2/32\pi^3(g_1\Delta_1^2 + g_2\Delta_2^2)$. If $\gamma \ll \alpha_{1,2}$, the gaps Δ_μ are decoupled from θ, so variation of Eq. (9) with respect to θ yields the sine-Gordon equation $L_\theta^2\nabla^2\theta = \pm\sin\theta$ which has a single-soliton solution [10], or periodic solutions analogous to the Josephson vortices in a magnetic field [13]. However, the interband phase difference θ is *decoupled* from static magnetic fields, while the Gibbs free energy of a Josephson contact contains the interaction term $\propto (\mathbf{H}\nabla\theta_{\mathbf{J}})$. Thus, equilibrium nonuniform solutions $\theta(x)$ are energetically unfavorable as compared to the uniform phase-locked state $\theta = 0$ for $\gamma < 0$ or $\theta = \pi$ for $\gamma > 0$. By contrast, various dynamic or quenched phase textures can be generated during current-induced interband breakdown.

We consider the weak coupling limit $\gamma \ll \alpha_\mu$, for which the uniform gaps $\Delta_{1,2}$ are unaffected by $\theta(\mathbf{r},t)$, so Eqs. (5)-(6) give a time-dependent London equation with the account of dissipation and charge imbalance effects [14]. Expressing $\mathbf{Q}_{1,2}$ in Eqs. (5) in terms of $\mathbf{J}$ and $\nabla\theta$, and then subtracting the equations for θ_1 and θ_2 from each other, we obtain the following equation for θ

$$\tau_\theta\dot{\theta} = L_\theta^2\nabla^2\theta \pm \sin\theta + \alpha_\theta\text{div}\mathbf{J}_s, \qquad (10)$$

where the relaxation time τ_θ, the decay length L_θ, and the charge coupling parameter α_θ are given by

$$\tau_\theta = \Gamma_1\Gamma_2\Delta_1\Delta_2/|\gamma|(\Gamma_1\Delta_1^2 + \Gamma_2\Delta_2^2), \qquad (11)$$

$$L_\theta^2 = g_1 g_2\Delta_1\Delta_2/|\gamma|(g_1\Delta_1^2 + g_2\Delta_2^2), \qquad (12)$$

$$\alpha_\theta = \frac{\phi_0\Delta_1\Delta_2(\Gamma_1 g_2 - \Gamma_2 g_1)}{4\pi c|\gamma|(g_1\Delta_1^2 + g_2\Delta_2^2)(\Gamma_1\Delta_1^2 + \Gamma_2\Delta_2^2)} \qquad (13)$$

The signs in Eq. (10) correspond to the sign of γ. As seen from Eq. (10), the phase mode is unaffected by any distribution of bulk supercurrents, unless there is a nonequilibrium charge imbalance $\text{div}\mathbf{J}_s = -\sigma\text{div}\mathbf{E}$ caused by the

electric field penetrating from normal current leads. The excess charge density provides a driving term in the sine-Gordon equation for θ, where the coupling constant α_θ is nonzero if $g_1\Gamma_2 \neq g_2\Gamma_1$, which implies different electron diffusivities in the bands 1 and 2.

To obtain the equation for $\mathbf{E}$, we add Eqs. (5) for θ_1 and θ_2, then take the gradient of the sum and express $\mathbf{Q}_{1,2}$ in terms of $\mathbf{J}$ and $\nabla\theta$. This yields

$$\tau_e\dot{\mathbf{E}} + \mathbf{E} - L_e^2\text{graddiv}\mathbf{E} + \alpha_e\nabla\dot{\theta} = \tau_e\dot{\mathbf{J}}/\sigma, \qquad (14)$$

where $\mathbf{J}(t)$ is the driving current density, L_e is the electric field penetration depth, τ_e is the charging time constant, and the coupling term $\alpha_e\nabla\dot{\theta}$ describes an electric field caused by moving phase textures:

$$\tau_e = \sigma\phi_0^2/8\pi^2 c^2(g_1\Delta_1^2 + g_2\Delta_2^2), \qquad (15)$$

$$L_e^2 = \sigma\phi_0^2/8\pi^2 c^2(\Gamma_1\Delta_1^2 + \Gamma_2\Delta_2^2), \qquad (16)$$

$$\alpha_e = 2|\gamma|\Delta_1\Delta_2\alpha_\theta \qquad (17)$$

We first use Eqs. (10) and (14) to calculate the phase textures in a microbridge of length $2a$ (Fig. 1a) for which Eq. (10) at $\gamma < 0$ takes the form

$$L_\theta^2\theta'' - \sin\theta + \alpha_\theta J_s' = 0 \qquad (18)$$

Here $\theta'(x)$ and $J_s(x)$ vanishes at the bridge edges, $x = \pm a$, where $E(x,t) = E_0\cosh(x/L_e)/\cosh(a/L_e)$ equals the value $E_0 = J/\sigma$ provided by an external power source, $\theta'(0) = E'(0) = 0$, and the prime denotes differentiation over x. Now we obtain the condition under which Eq. (18) has a stable solution $\theta(x,\theta_a)$ localized at the edge of a long $(a \gg L_\theta)$ bridge, where $\theta_a = \theta(a)$. We first consider the limit $L_\theta \gg L_e$, for which $J_s'(x) = -\sigma E'(x)$ is essential only in a narrow region $a - L_e < x < a$. Multiplying Eq. (18) by θ' and integrating from 0 to $a - L_e$, we obtain $\theta'(a - L_e) = -2\sin\theta_a/2$. Here $\theta'(a - L_e) = -\alpha_\theta J/L_\theta^2$ is obtained by integrating Eq. (18) over $a - L_e < x < a$, where $\theta(x - L_e) \approx \theta_a$, and $\sin\theta$ can be neglected. Excluding $\theta'(a - L_e)$, we arrive at the equation $\alpha_\theta J = 2L_\theta|\sin\theta_a/2|$, which has solutions for θ_a only below the critical current density $J_t = 2L_\theta/\alpha_\theta$:

$$J_t = \frac{8\pi c(\Gamma_1\Delta_1^2 + \Gamma_2\Delta_2^2)\sqrt{|\gamma|g_1 g_2(g_1\Delta_1^2 + g_2\Delta_2^2)}}{\phi_0\sqrt{\Delta_1\Delta_2}|g_1\Gamma_2 - g_2\Gamma_1|} \qquad (19)$$

In the opposite limit $L_\theta \ll L_e$, the Laplacian in Eq. (10) can be neglected. Then the steady-state solutions $\sin\theta(x) = -\sigma\alpha_e E'(x)$, and $E(x) = E_0\cosh(x/L_e)/\cosh(a/L_e)$, exist only if $\sigma\alpha_e E'(a) < 1$, or $J < J_t = L_e/\alpha_e\tanh(a/L_e)$, where

$$J_t = \frac{|\gamma|[\sigma(\Gamma_1\Delta_1^2 + \Gamma_2\Delta_2^2)]^{1/2}(g_1\Delta_1^2 + g_2\Delta_2^2)}{\sqrt{2}\Delta_1\Delta_2|g_1\Gamma_2 - g_2\Gamma_1|\tanh(a/L_e)} \qquad (20)$$

If $\gamma(T)$ and $\alpha_\mu(T)$ linearly pass through zero at T_c, Eqs. (19) and (20) yield $J_t \propto (T_c - T)^{3/2}$.

For $J > J_t$, the charge-induced interband breakdown gives rise to a striking dynamic phase texture in which solitons periodically appear near the current leads and then propagate to the bulk. The actual soliton dynamics depends on the particular boundary conditions, so we performed numerical simulations of Eqs. (10)-(17) for the different geometries shown in Fig. 1. We consider the limit $\alpha_e \alpha_\theta \ll \min(\tau_\theta L_e^2, \tau_e L_\theta^2)$ of weak coupling between θ and E, for which the driving term div$\mathbf{E}$ in Eq. (10) is mostly determined by the *static* $E(x)$ near the current lead, while the last term in Eq. (17) gives only a small ac correction to E of the second order in α_e[15].

Fig. 2 shows the soliton formation near the current lead in the thin microbridge (Fig 1a) as J was instantaneously turned on from zero to a value below J_t. Such current step produces a phase soliton localized near the edge, while the bulk of the bridge remains in the phase-locked state with either $\theta = 0$ or π, depending on the sign of γ. For $J < J_t$, this behavior is characteristic of all geometries shown in Fig. 1, whereas the dynamic phase texture for $J > J_t$ can be very different.

We start with the bridge geometry (Fig. 1a), for which $E(x,t)$ and $\theta(x,t)$ are even and odd functions of x, respectively, so the boundary conditions are: $E(\pm a, t) = E_0$, $E'(0,t) = 0$, $\theta(0) = 0$, and $\theta'(\pm a, t) = 0$. The latter condition ensures that supercurrents in both bands vanish at the normal electrodes, where $J = \sigma E_0$. In this case a soliton first appears at the bridge edge, but for $J > J_t$, it is pushed to the bulk by the strong gradient of $E(x)$. Then the next soliton forms near the edge and the process repeats periodically, resulting in the propagation of the soliton chain into the bulk, as shown in Fig. 3. After the first soliton reaches the center of the bridge, it stops (because $\theta(0) = 0$), while new solitons keep entering the bridge from the current lead. During this soliton pileup, the mean slope $\bar{\theta}'(t)$ increases, reaching a critical value $\bar{\theta}'$ at which the soliton generation at the edge stops and a static phase texture forms. For $J \gg J_c$, the maximum soliton density $n = \bar{\theta}'_c/2\pi$ can be estimated from the static Eq. (10) in which the rapidly oscillating $\sin\theta$ can be neglected, and $E(x) = J\sinh(x/L_e)/\sigma\sinh(a/L_e)$. This yields $\theta'(x) = \alpha_\theta J L_e[\cosh(a/L_e) - \cosh(x/L_e)]/L_\theta^2 \sinh(a/L_e)$, giving for the long bridge ($a \gg L_e$):

$$\bar{\theta}'_c = J L_e \phi_0 (\Gamma_1 g_2 - \Gamma_2 g_1)/4\pi c g_1 g_2 (\Gamma_1 \Delta_1^2 + \Gamma_2 \Delta_2^2) \quad (21)$$

During the formation of the soliton chain, $t < t_c \sim \tau_\theta a \bar{\theta}'_c/2\pi$, voltage oscillations are generated on the bridge. A similar transient behavior occurs at the point contact (Fig. 1b), for which div$\mathbf{E} = 2I\exp(-r/L_e)r_0/\pi L_e(r_0 + L_e)r$, where I is the current through the semi-spherical contact of radius r_0. In this case the increase of I above I_c causes propagation of concentric soliton shells until the static structure forms as the critical phase gradient $\sim \theta'_c$ is reached.

A very different soliton dynamics occurs in the 4-terminal geometry (Fig. 1c), for which the supercurrents make 90° turns around the central stagnation point ($x = 0$) where $\nabla\theta = 0$. As a result, $E(x)$ and $\theta(x,t)$ become odd and even functions of x, respectively, which radically changes the dynamics of the phase textures as compared to the bridge, since now the solitons do not stop as they reach the center, but keep moving until they reach the opposite current lead. This is due to the fact that the driving charge density div$\mathbf{E}$ does not change sign all the way along the horizontal leg of the cross in Fig. 1c. The corresponding total extra charge (which plays the same role here as the transport current in a Josephson junction) is exactly compensated by the opposite charge distributed along the vertical leg. Thus, the mean phase gradient $\bar{\theta}'$ remains constant, while the phase slippage at the normal leads results in a continuous soliton shuttle and periodic voltage oscillations between the current lead and the center as shown in Fig. 4. For $L_e \gg L_\theta$, the ac voltage $V_\omega(t) = \sum_{m=1}^\infty V_m \cos(m\omega t + \psi_m)$ can be estimated by integrating Eq. (14) from 0 to a, which yields $\tau_e \dot{V}_\omega + V_\omega = \alpha_e[\dot{\theta}_a(t) - \dot{\theta}(0,t)]$. Here the values $\theta_a(t) \sim \theta(0,t)$ obey the RSJ equation $\tau_\theta \dot{\theta}_a + \sin\theta_a = \beta$, for which $\tau_\theta \dot{\theta}_a = \sqrt{\beta^2 - 1}[1 + 2\sum_{m=1}^\infty (\beta - \sqrt{\beta^2 - 1})^m \cos m\omega t]$, $\beta = J/J_t$, and $\omega = \tau_\theta^{-1}\sqrt{\beta^2 - 1}$ [16]. As a result, we obtain $\tan\psi_m = m\omega\tau_e$, and

$$V_m \simeq \frac{2\alpha_e}{\tau_\theta}\sqrt{\beta^2 - 1}\frac{(\beta - \sqrt{\beta^2 - 1})^m}{\sqrt{1 + (m\omega\tau_e)^2}}. \quad (22)$$

$V_m(\beta)$ is maximum at $\beta = \beta_m$, where $\beta_m \simeq 1 + 0.5(\tau_\theta/m\tau_e)^{2/3}$ for $\tau_\theta \ll \tau_e$. Thus, a two-gap *superconducting* bridge between normal banks exhibits the behavior of an overdamped Josephson junction. If the widths of the superconducting legs in Fig. 1c are different, either a unidirectional and mutually orthogonal soliton motion may occur, if $J > J_t$ in one or both legs, respectively.

The soft interband phase mode could also manifest itself in rf absorption at frequencies below the small gap Δ_2. Interaction of the θ-mode with the rf electric field depends on the polarization of $\mathbf{E}$: if $\mathbf{E}(t)$ is parallel to the sample surface, then div$\mathbf{E} = 0$, so the phase mode is not excited by the rf field. If, however the rf field has a component perpendicular to the sample surface, then the phase mode contributes to the rf impedance.

In conclusion, we predict an interband breakdown induced by nonequilibrium quasiparticles in two-gap superconductors. It results in spontaneous generation of static and/or dynamic phase textures, causing the Josephson voltage oscillations. The observation of these oscillations would unambiguously indicate the multicomponent nature of the superconducting order parameter.

FIG. 1: Geometries in which the interband phase breakdown could occur. Here N labels normal electrodes, gray domains show phase solitons moving along thin arrows, and block arrows indicate current directions. Static phase textures form in microbridges (a) and point contacts (b), while in the four-terminal geometry (c) the solitons continuously move from one current lead to another.

FIG. 2: Formation of a static phase soliton in $\theta(x)$ near the bridge edge after the current density was instantaneously turned on from 0 to $J = 0.99J_t$ at $t = 0$. Times and distances from the center ($x = 0$) are taken in the units of τ_θ and a, respectively, $L_e = a/10$, $L_\theta = 0.1L_e$.

FIG. 3: Dynamics of formation of a static soliton chain in the bridge of length 2a after $J(t)$ was instantaneously turned on from 0 to $1.025J_t$ at $t = 0$. Only the right half ($0 < x < a$) is shown, and the rest is the same as in Fig. 2.

FIG. 4: Moving soliton shuttle along the right half of the horizontal leg ($0 < x < a$) in the four-terminal geometry shown in Fig. 1c. $J(t)$ was instantaneously turned on from 0 to $1.012J_t$ at $t = 0$, and the rest is the same as in Fig. 2.

[1] A. Liu, I.I. Mazin and J. Kortus, Phys. Rev. Lett. **87** 087005 (2001).

[2] G. Karapetrov, *et al.,* Phys. Rev. Lett. **86**, 4374 (2001); F. Giubileo *et al., ibid.* **87**, 17708 (2001); H. Schmidt *et al., ibid.* **88**, 127002 (2002).

[3] T. Yokoya *et al.,* Science **294**, 2518 (2001).

[4] R. Joynt and L. Tallifer, Rev. Mod. Phys. **74**, 235 (2002).

[5] D. Jerome and H.J. Schulz, Adv. Phys. **31**, 299 (1982).

[6] J. Singleton and C. Mielke, Contemporary Phys. **43**, 63 (2002).

[7] A.J. Legget, Prog. Theor. Phys. **36**, 901 (1966); Rev. Mod. Phys. **47**, 331 (1975).

[8] M. Sigrist and K. Ueda, Rev. Mod. Phys. **63**, 239 (1991).

[9] D.F. Agterberg, E. Demler, and B. Janko, cond-mat/0201376v1.

[10] Y. Tanaka, Phys. Rev. Lett. **88**, 017002-1 (2002).

[11] N. Kopnin, *Theory of Nonequilibrium Superconductivity.*(Oxford University Press, New York, 2001).

[12] H. Suhl, B.T. Matthias, and L.R. Walker, Phys. Rev. Lett. **12**, 552 (1959).

[13] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).

[14] If Δ_1 appears at a higher T_{c1}, the induced gap $\Delta_2 = \gamma\Delta_1 \cos\theta/\alpha_2$ at $T_{c2} < T < T_{c1}$ depends on θ. In that case θ is a poor variable, because Eq. (4) turns into a single TDGL equation for Δ_1 with $\alpha_1 \rightarrow \alpha_1 - \gamma^2 \cos^2\theta/\alpha_2$, which describes only locked states with $\theta = 0$ or π.

[15] For $\tau_e \gg \tau_\theta$ and $L_e^2 \gg L_\theta^2$, the ac component of E obeys $\tau_e \dot{E} = L_e^2 E'' + \alpha_e \dot\theta'$, whence $E' \simeq -\alpha_e \dot\theta/L_e^2$ for small soliton velocities $v \ll L_e/\tau_e$, and $E' \simeq \alpha_e \theta''/\tau_e$ for $v \gg L_e/\tau_e$. This E' just renormalizes the length and time scales in Eq. (10): $\tau_\theta \rightarrow \tau_\theta + \alpha_e\alpha_\theta/L_e^2$, $L_\theta \rightarrow L_\theta$ for $v \ll L_e/\tau_e$, and $\tau_\theta \rightarrow \tau_\theta$, $L_\theta^2 \rightarrow L_\theta^2 + \alpha_e\alpha_\theta/\tau_\theta$ for $v \gg L_e/\tau_e$, but does not change the behavior of $\theta(x,t)$.

[16] L.G. Aslamazov and A.I. Larkin, JETP Lett. **9**, 87 (1969).